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Onset of linear instability in homogeneous plasmas

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Abstract. We consider the linear instability in homogeneous plasmas that occurs when the effective one-dimensional distribution of electrons is not a 'single maximum' function. We construct a quadratic form $[\,,\,]_R$ to explain why no linear instability occurs for sufficiently high wavenumbers k, even though the energy quadratic form is indefinite for all k. $[\,,\,]_R$ is conserved in the linearized evolution, and positive-definite precisely up to the wavenumber that the instability actually occurs. We argue against a 'structural instability' due to nonlinear terms.

1. Introduction

A basic problem in plasmas is the study of self-consistent small perturbations of the electron density and the electric field on a homogeneous equilibrium. Landau [1] demonstrated the collisionless damping of perturbations for stable equilibrium distribution functions which are analytic in the complex velocity plane. The analysis has been extended to general equilibrium functions ([2, 3] and more recently [4]).

The perturbation may be decomposed into Fourier modes. The equilibrium as well as the perturbation distributions may be integrated in transverse velocities to give effective distribution functions $g_0(v)$, $g(v)$ respectively, for the velocity component along the wavevector direction. In the case that $g_0(v)$ has a single maximum in v stability of the plasma can be demonstrated for all Fourier modes in terms of the positivity of the energy for all 'allowed' perturbations [5, 2]. These are perturbations that can be generated by the application of canonical transformations to the equilibrium configuration. (This set includes all perturbations that can be generated by external electric fields acting on the equilibrium configuration, in the direction of the wavevector.)

If $g_0(v)$ has more than one maximum, then for *every* Fourier mode k there are allowed perturbations that lower the energy of the configuration, or keep the energy unchanged so that the perturbation can grow arbitrarily large from the viewpoint of linear theory. At first one might think that there must be a corresponding linear instability, for all k, for such equilibrium configurations. But analysing the dispersion relation for the linear evolution reveals that no such instability occurs for wavenumbers k outside a range (k_0, k_0'), for some k_0, k_0' depending on the equilibrium configuration. How do we understand this 'anomalous stability'? This problem is important if we are to study recent conjectures that even for k outside the above range there is a 'structural instability' in the configuration which can be triggered by a small change in the evolution equation, e.g. by the inclusion of nonlinear terms [6]. The situation has been compared

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[6] to that for the Hamiltonian

$$H = -\tfrac{1}{2}\omega_1(p_1^2+q_1^2)+\tfrac{1}{2}\omega_2(p_2^2+q_2^2). \tag{1.1}$$

Solutions for H are purely oscillatory, but if $\omega_2/\omega_1=2$ for example, then the addition of small nonlinear terms like $\alpha(2q_1p_1p_2-q_2(q_1^2-p_1^2))$ gives explosive solutions (which diverge in a finite time) that grow by transferring energy from the negative energy mode to the positive energy mode [7].

In this paper we ask the question: What characterizes the wavenumbers k_0, k_0' at which the linear instability sets in? These are not the points where the energy quadratic form (which we denote by $[\,,\,]_Q$) becomes of indefinite sign, since as mentioned above the energy form is indefinite for *all* k for these unstable equilibrium configurations. We construct another quadratic form $[h, g]_R$ which has the following properties: (*a*) $[\,,\,]_R$ is conserved in the (linearized) evolution; and (*b*) $[\,,\,]_R$ is positive-definite on the space of all perturbations with wavenumber k outside the range (k_0, k_0'), while it ceases to be positive-definite for $k=k_0, k_0'$. $[\,,\,]_R$ is the simplest of a family of quadratic forms sharing the properties (*a*) and (*b*). No quadratic form can be constructed which commutes with the evolution and is positive-definite for $k=k_0, k_0'$; thus these invariants do characterize the onset of linear instability.

We follow with some remarks for the nonlinear case. Stellar dynamics is governed by the collisionless Boltzmann equation in the same manner as plasmas, though with the further complication that the equilibrium configuration is inhomogeneous, so that a Fourier transformation cannot separate modes. The 'anomalous stability' mentioned above has its counterpart for stellar dynamics, and a corresponding structural instability might be expected [8]. But numerical simulations of the one-dimensional gravitational N-body problem indicate instead a robust stability for such configurations [9]. This result might be explained [10] by an application of the Kolmogorov–Arnold–Moser (KAM) theorem of classical mechanics [11]. Similar considerations are expected to apply to plasmas, where the above mentioned family of quadratic forms can serve as the integrals for the unperturbed Hamiltonian in the KAM analysis.

The plan of this paper is as follows. Section 2 details the basic equations for the linear evolution, the dispersion relation and the energy quadratic form. Section 3 presents the quadratic form $[\,,\,]_R$ and establishes its properties. Section 4 discusses the nonlinear case. Section 5 concludes with some comments on the mathematical nature of the onset of the linear instability.

2. Linearized evolution and the energy form

Let $Nf(\boldsymbol{x}, \boldsymbol{v}, t)$ be the density of electrons in the six-dimensional position–velocity space. The self-consistent oscillations of the electron density and the electric field are described by the collisionless Boltzmann equation

$$f_{,t}+\boldsymbol{v}\cdot f_{,\boldsymbol{x}}-\boldsymbol{E}\cdot\frac{e}{m}f_{,\boldsymbol{v}}=0. \tag{2.1}$$

where

$$\boldsymbol{E}(\boldsymbol{x}, t)=-eN\int\frac{(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}f(\boldsymbol{x}', \boldsymbol{v}', t)\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{v}' \tag{2.2}$$

For small perturbations on a homogeneous equilibrium configuration we write $f(\boldsymbol{x}, \boldsymbol{v}, t) = f_0(\boldsymbol{v}) + f_1(\boldsymbol{x}, \boldsymbol{v}, t)$, to obtain

$$f_{1,t} + \boldsymbol{v} \cdot f_{1,x} + \frac{e^2 N}{m} f_{0,v} \cdot \int \frac{(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} f_1(\boldsymbol{x}', \boldsymbol{v}', t)\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{v}' = 0. \tag{2.3}$$

f_1 may be Fourier transformed in position space:

$$\tilde{f}_1(\boldsymbol{k}, \boldsymbol{v}, t) = \int f_1(\boldsymbol{x}, \boldsymbol{v}, t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}. \tag{2.4}$$

Conservation of total charge implies that we may restrict to $\boldsymbol{k} \neq 0$. Fix $\boldsymbol{k}$, and let the x-axis be parallel to $\boldsymbol{k}$. Define

$$g(k, v) = \int \tilde{f}_1(k, v, v_y, v_z)\,\mathrm{d}v_y\,\mathrm{d}v_z. \tag{2.5}$$

We will write $g(v)$ for $g(k, v)$, as k remains unchanged in any given equation. Then (2.3) reduces to

$$((\mathrm{i}k)^{-1}\partial_t + v)g(v, t) = \frac{\omega_P^2}{k^2} g_{0,v}(v) \int_{-\infty}^{\infty} g(v', t)\,\mathrm{d}v' \tag{2.6}$$

where $\omega_P = (4\pi e^2 N/m)^{1/2}$ is the plasma frequency and

$$g_0(v) = \int f_0(v, v_y, v_z)\,\mathrm{d}v_y\,\mathrm{d}v_z. \tag{2.7}$$

If there exists a well-defined oscillation mode with frequency ω:

$$g(v, t) = g(v)\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{2.8}$$

then we get

$$(-\omega/k + v)g(v) = \frac{\omega_P^2}{k^2} g_{0,v}(v) \int_{-\infty}^{\infty} g(v')\,\mathrm{d}v'. \tag{2.9}$$

Integrating (2.9) in v we get the dispersion relation for modes

$$\int \frac{g_{0,v}(v)}{(v-\omega/k)}\,\mathrm{d}v = \frac{k^2}{\omega_P^2}. \tag{2.10}$$

As mentioned in section 1, one restricts to perturbations that can be produced by a canonical transformation on the equilibrium configuration. The energy (per unit transverse area) of such a perturbation is proportional, to lower order, to $[g, g]_Q$ where $[\ ,\ ]_Q$ is the quadratic form [5]

$$[h, g]_Q = \int \frac{h^*(x, v)g(x, v)v}{-g_{0,v}(v)}\,\mathrm{d}x\,\mathrm{d}v + \frac{\omega_P^2}{2} \int h^*(x', v')|x-x'|^{-1} g(x, v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}x'\,\mathrm{d}v'. \tag{2.11}$$

Reducing to Fourier components we find that (2.11) vanishes for h, g having different wavenumbers, while for both having wavenumber k we get

$$[h, g]_Q = \int \frac{h^*(v)g(v)v}{-g_{0,v}(v)}\,\mathrm{d}v + \frac{\omega_P^2}{k^2}\left(\int h(v')\,\mathrm{d}v'\right)^* \left(\int g(v)\,\mathrm{d}v\right) \tag{2.12}$$

for the contribution to (2.11) per unit x length.

3. The quadratic form $[\,,\,]_R$ and its properties

The first term in the energy quadratic form $[g, g]_Q$ (equation (2.12)) has the form $\int \alpha(v)|g(v)|^2\,\mathrm{d}v$ where $\alpha(v)=-v/g_{0,v}(v)$. We distinguish two kinds of equilibrium configurations:

(*a*) $g_{0,v}=0$ for $v=0$. This class includes the distributions $g_0(v)=\tilde{g}_0(v^2)\equiv\tilde{g}_0(E)$. Let $g_{0,V}=0$ in addition for $v=v_i$, $i=1,\ldots,n$†. Then $\alpha(v)$ changes sign at the v_i, passing through infinity at the change. $\alpha(v)$ does not change sign at $v=0$.

(*b*) $g_{0,v}\neq 0$ for $v=0$. Then in addition to the changes in sign at the v_i, $\alpha(v)$ changes sign at $v=0$, but this change occurs by passing through the value zero, not infinity.

For equilibria of type (*a*)

$$|\alpha(v)|>C>0 \tag{3.1}$$

for some constant C, while such an inequality does not hold for equilibria of type (*b*). We will investigate only the type (*a*) configurations, except for some brief remarks at the end of this section about type (*b*).

Let

$$Lg(x,v)=-\mathrm{i}v\partial_x g(x,v)-\frac{\mathrm{i}\omega_P^2}{2}g_{0,v}(v)\int \mathrm{d}x'\,\mathrm{d}v'\,\varepsilon(x-x')g(x,v') \tag{3.2}$$

where $\varepsilon=1$ for positive argument, -1 for negative argument. Thus on the space of Fourier component k functions

$$Lg(v)=kvg(v)-\frac{\omega_P^2}{k}g_{0,v}(v)\int \mathrm{d}v'\,g(v'). \tag{3.3}$$

We have from (2.6)

$$\mathrm{i}\partial_t g=Lg. \tag{3.4}$$

For equilibria of type (*a*) we define (for h, g, functions of x, v)

$$[h,g]_R=\left[h,\prod_{i=1}^{n}(L+\mathrm{i}v_i\partial_x)g\right]Q. \tag{3.5}$$

Reducing to Fourier components we find that (3.5) vanishes for h, g having different wavenumbers, while for both having wavenumber k we get

$$[h,g]_R=\left[h,\prod_{i=1}^{n}(L-kv_i)g\right]_Q. \tag{3.6}$$

It is readily checked that

$$[h,Lg]_Q=[g,Lh]_Q^* \tag{3.7}$$

from which follows that

$$[h,g]_R=[g,h]_R^* \tag{3.8}$$

so that $[\,,\,]_R$ is symmetric. From (3.4) follows that $[\,,\,]_R$ is conserved in the evolution.

Defining

$$Qg(v)=\frac{vg(v)}{-g_{0,v}(v)}+\frac{\omega_P^2}{k^2}\left(\int g(v')\,\mathrm{d}v'\right) \tag{3.9}$$

† We assume for simplicity that these zeros are simple. All arguments for multiple zeros follow by taking the limit of coinciding simple zeros.

we have

$$[h, g]_Q = [h, Qg]_{L^2} \tag{3.10}$$

where

$$[g_1, g_2]_{L^2} = \int g_1^*(v) g_2(v)\, dv \tag{3.11}$$

is the L^2 inner product on the space of functions of v. Thus

$$[h, g]_R = \left[h, Q \prod_{i=1}^{n} (L - kv_i) g \right]_{L^2}. \tag{3.12}$$

The allowed perturbations $g(x, v)$ have the form ($\{ , \}$ is the Poisson bracket)

$$\begin{aligned} g(x, v) = \{J(x, v), g_0(v)\} = J_{,x}(x, v) g_{0,v}(v) & \\ \to C_i(v - v_i) \qquad & v \to v_i \\ \to C'v \qquad & v \to 0. \end{aligned} \tag{3.13}$$

We will assume that

$$v^{n+3} g_{0,v}(v) \to 0 \qquad \text{as } |v| \to \infty \tag{3.14}$$

so that $[,]_R$ is bounded on all such perturbations†. The space of 'good' perturbations we denote by $\mathscr{S}$; it is contained in $L^2(v)$.

We first motivate the form $[,]_R$ by taking a simple distribution $g_0(v)$ and establishing that there exists a $\hat{k}_0$ such that for $|k| > \hat{k}_0$, $[,]_R$ is positive definite. Let $g_0(v) = \tilde{g}_0(v^2) = \tilde{g}_0(E)$, and

$$\begin{aligned} \tilde{g}_0'(E) < 0 \qquad & E > E_1 \\ \tilde{g}_0'(E) > 0 \qquad & E < E_1 \end{aligned} \tag{3.15}$$

where $\tilde{g}_0'(E) = d\tilde{g}_0(E)/dE$. Thus $g_0(v)$ decreases with $|v|$ for $|v| > v_1 = \sqrt{2E_1}$, but increases with $|v|$ for $0 < |v| < v_1$, and we have a 'two maxima distribution'. In the limit $|k| \to \infty$, we get $L \to kv$, and

$$[g, g]_R \to \int \frac{k^2 v (v^2 - v_1^2)}{-g_{0,v}(v)} |g(v)|^2\, dv. \tag{3.16}$$

We note that

$$g_{0,v} = -v(v^2 - v_1^2)\hat{g}_0(v) \tag{3.17}$$

with $\hat{g}_0(v)$ bounded and positive in the support of $g_0(v)$. Thus we find that at least the limit (3.16) is positive-definite. Further,

$$L^2 g(v) = k^2 v^2 g(v) - v\omega_P^2 g_{0,v}(v) \int g(v')\, dv' - \omega_P^2 g_{0,v}(v) \int v' g(v')\, dv'. \tag{3.18}$$

† $J(x, v)$ is assumed to be a bounded square-integrable function of x, v with at least n continuous derivatives in x to permit defining $[,]_R$. It is possible to generate growing modes even on 'single maximum' distributions $g_0(v)$ with sufficiently singular perturbations [2]; we will avoid investigating such possibilities here.

Using (2.11) this yields

$$[g,g]_R = k^2 \int (\hat{g}_0(v))^{-1}|g(v)|^2\,\mathrm{d}v + 2\omega_P^2\,\mathrm{Re}\left\{\left(\int v^2 g^*(v)\,\mathrm{d}v\right)\left(\int g(v')\,\mathrm{d}v'\right)\right\}$$
$$+\omega_P^2\left|\int v g(v)\,\mathrm{d}v\right|^2 + \left(-\omega_P^2 v_1^2 + \frac{\omega_P^4}{k^2}\right)\left|\int g(v)\,\mathrm{d}v\right|^2. \tag{3.19}$$

where we have assumed the normalization

$$\int g_0(v)\,\mathrm{d}v = 1. \tag{3.20}$$

We use the Schwarz inequality for the non-positive-definite terms in (3.19). From (3.17) and (3.20) follow that

$$\int \hat{g}_0(v)\,\mathrm{d}v \equiv D_1^2 < \infty \tag{3.21}$$

$$\int v^4 \hat{g}_0(v)\,\mathrm{d}v \equiv D_2^2 < \infty. \tag{3.22}$$

Then

$$\left|\int g(v)\,\mathrm{d}v\right| \leqslant \left(\int |g(v)|^2(\hat{g}_0))^{-1}\,\mathrm{d}v\right)^{1/2} D_1 \tag{3.23}$$

$$\left|\int v^2 g(v)\,\mathrm{d}v\right| \leqslant \left(\int |g(v)|^2(\hat{g}_0(v))^{-1}\,\mathrm{d}v\right)^{1/2} D_2. \tag{3.24}$$

So for $k^2 > \omega_P^2(v_1^2 D_1^2 + 2D_1D_2)$, $[g,g]_R$ is positive. A similar argument holds for any distribution of type (a), and it follows that $[\ ,\ ]_R$ is positive-definite on perturbations of sufficiently large wavenumber $|k|$.

Our goal is to establish:

Theorem 1. The quadratic form $[\ ,\ ]_R$ is positive-definite for sufficiently large absolute values of k. As k is moved in continuously (from plus or minus infinity) $[\ ,\ ]_R$ ceases to be positive-definite exactly at the point where (2.10) has a solution.

The nature of solutions to (2.10) is readily analysed [2, 4]. There are no solutions for sufficiently large $|k|$. As k moves in from infinity, a solution with real ω can appear, with the integral in (2.10) considered as a principal value integral. This solution does not correspond to a real frequency oscillation mode however, because the corresponding eigenfunction from (2.9) is singular. As k is moved further, two such real values of ω can 'collide'. At this point we get an oscillatory solution to (3.4), and a linearly growing solution. Further changing k causes this double point to split into a pair of complex conjugate eigenvalues, giving exponentially growing and exponentially decaying solutions to (3.4). Alternatively, there may be no 'collision' but a real solution point branches into three, one remaining real, while the other two leaving the real line to form a complex-conjugate pair.

We have the simple observation:

Lemma 2. If a pair of complex conjugate eigenvalues emerges from the real line at $\omega = k\hat{v}$ then we must have $g_{0,v}(\hat{v}) = 0$.

Proof. For $\omega = \omega_R + i\omega_I$, $\omega_I \to 0^+$, (2.10) gives

$$\int \frac{g_{0,v}(v)}{(v-\omega_R/k - i\omega_I/k)}\,dv$$
$$= \int_P \frac{g_{0,v}(v)}{(v-\omega_R/k)}\,dv + i\pi \int \delta(v-\omega_R/k) g_{0,v}(v)\,dv$$
$$= \int_P \frac{g_{0,v}(v)}{(v-\omega_R/k)}\,dv + i\pi g_{0,v}(\omega_R/k) = \frac{k^2}{\omega_P^2} \qquad (3.25)$$

where $\int_P$ is a principal value integral. The imaginary part of this equation establishes the Lemma. □

Thus the unstable mode can develop, for type (*a*) distributions, at $\omega = 0$ or at $\omega = kv_i$, $i = 1 \ldots n$. To relate this property of the dispersion relation to $[\,,\,]_R$ we have

Lemma 3. (*a*) The relation $Lg = 0$ is equivalent to $Qg = 0$, which implies $[g, g]_R = 0$.
(*b*) The relation $(L - kv_i)g = 0$ implies $[g, g]_R = 0$.

Proof. (*a*) $Lg = 0$ implies $g(v) = (\text{Constant})\; g_{0,v}(v)/v$, which is a regular function for type (*a*) distributions, and clearly solves $Qg = 0$. The converse is immediate too. $[g, g]_R = (-k)^n (\Pi_{i=1}^n (v_i))[g, g]_Q = 0$ from (3.10).
(*b*) This is obvious from the form of $[\,,\,]_R$. □

Lemmas 2 and 3 establish that $[\,,\,]_R$ cannot be positive-definite at the value of k at which an eigenmode arises for the evolution, as given by the dispersion relation (2.10). We now argue the converse part of theorem 1, i..e that $[\,,\,]_R$ remains positive-definite to the point that an unstable mode appears. As noted above, $[\,,\,]_R$ is positive-definite for sufficiently large $|k|$. As k moves in from say $-\infty$, let k_0 be the first point where $[\,,\,]_R$ is not positive-definite. We have

Lemma 4. At $k = k_0$, $[\,,\,]_R$ is positive semi-definite. There exist (for $k = k_0$) a finite non-zero number of functions $g_m(v)$ in the space $\mathscr{S}$ of allowed perturbations such that $[g_m, g_m]_R = 0$. Further, $Q\,\Pi_{i=1}^n (L - k_0 v_i) g_m \equiv T g_m = 0$ for each g_m.

Proof. First we prove the lemma for the case that $g_0(v)$ has compact support in v ($g_0(v) = 0$ for $|v|$ sufficiently large); next we indicate the modifications necessary for general $g_0(v)$. For the compact support case, T is a symmetric bounded operator on $L^2(v)$. From (3.6), $[h, g]_R = [h, Tg]_{L^2}$. Further, $T = M + N$, where

$$Mg(v) = \frac{vk^n}{-g_{0,v}(v)} \prod_{i=1}^n (v - v_i) g(v) \qquad (3.26)$$

is a symmetric positive-definite multiplicative operator and N has the form

$$Ng(v) = \sum_{i,j=1}^m N_{ij} l_i(v) \int l_j^*(v') g(v')\,dv' \qquad N_{ij} = N_{ji}^* \qquad (3.27)$$

with $m < 2^n$ and l_i known L^2 functions of v†. N is a symmetric bounded operator on $L^2(v)$ having finite rank (i.e. finite-dimensional range). Thus N is non-zero on the

† $l_i(v)$ can be taken to vanish outside the support of $g_0(v)$, which by (3.13) is also the support of the $g(v)$.

finite-dimensional span W of the l_i, and zero on the finite-dimensional span W of the l_i, and zero on the orthogonal complement $W^{\perp}$ of W in the L^2 norm. Thus if $[\,,\,]_R$ is not positive-definite in the space of perturbations then $[g, Tg]_{L^2}$ must be non-positive for some g in S and therefore in $L^2(v)$. The space W and the action of T on W can be seen to be continuous in k. Thus if $k=k_0$ is the first point where $[\,,\,]_R$ is not positive-definite then we have that $[g, g]_R \geqslant 0$ and $[g_m, g_m]_R=[g_m, Tg_m]_{L^2}=0$ for g_m in a subspace of the finite-dimensional space W. From these last two relations and the symmetry of $\bar{T}$ follows that $Tg_m=0$. From the form of $\bar{T}$ we find that each solution g_m actually lies in $\mathscr{S}$.

If $g_0(v)$ does not have compact support in v the functions $l_i(v)$ in (3.27) may not be in $L^2(v)$. Recalling (3.14) we define another inner product

$$[h, g]_A=\int h^*(v)g(v)A(v)\,\mathrm{d}v \tag{3.28}$$

where $A(v)=(v^{2n+2}+1)$. Thus for g in $\mathscr{S}$, the space of allowed perturbations, $[g, g]_A<\infty$. We can write $[h, g]_R=[h, (\tilde{M}+\tilde{N})g]_A$ where $\tilde{M}=(A(v))^{-1}M$ with M as in (3.26), and $\tilde{N}g(v)=\int K_N(v, v')A(v')g(v')\,\mathrm{d}v'$. Here $K_N(v, v')$ is the kernel

$$K_N(v, v')=(A(v))^{-1}\sum_{i,j=1}^{m} N_{ij}l_j(v)l_i^*(v')(A(v'))^{-1}. \tag{3.29}$$

It can be checked that the functions $l_i(v)(A(v))^{-1}$ have finite A-norm, so that K_N describes a finite rank operator in the Hilbert space of functions with the inner product $[\,,\,]_A$, which contains $\mathscr{S}$. The rest of the argument proceeds as before. □

Thus at $k=k_0$, $Tg=Q\,\Pi_{i=1}^n(L-kv_i)\;g=0$ for at least one g in the space of perturbations. Either $\Pi_{i=1}^n(L-kv_i)g\equiv g_1=0$ or g_1 solves $Qg_1=0$. By lemma 3, part (a), the latter possibility implies that $Lg_1=0$, which implies $Lg=0$. If $g_1=0$, then $(L-kv_i)g=0$ for some v_j, and we again have that g is an eigenfunction of L. We have therefore established the other half of theorem 1, i.e. that $[\,,\,]_R$ is positive-definite until $k=k_0$ when a solution appears for the dispersion relation (2.10). A similar argument holds when k is decreased from ∞ to k_0'. □

Our result may be interpreted as saying that for k outside the range (k_0, k_0') the evolution operator L is a self-adjoint operator on a positive-definite Hilbert space of perturbations, but $[\,,\,]_Q$ does not provide the positive-definite inner product. If $D(x)$ is any positive function of its argument, bounded below, then $[h, D(L)\,\Pi_{i=1}^n(L-kv_i)g]_Q$ is also a conserved positive-definite form. $[\,,\,]_R$ is the simplest of the family of quadratic forms having these properties.

If ω is a complex eigenvalue of L then $[g, Lg]_Q=\omega[g, g]_Q$. But by (3.7) the LHS is real, so that $[g, g]_Q$ (which is also real) must vanish. The limit of this result for ω at the real axis yields that any conserved quadratic form $[g, B(L)_g]_Q$, (B an arbitrary bounded function) must be null for some g at the onset of instability, and so cannot be positive-definite. Alternatively, one notes that there is a two-dimensional subspace of functions associated to the pair of complex conjugate eigenvalues ω, ω^*. But there is only one eigenfunction when ω is taken to the real line, because (2.9) has a unique solution for any ω. The other vector at this point grows linearly in time t, so that no positive-definite quadratic form can be conserved. Thus we see that it is not possible to construct conserved quadratic forms that remain positive-definite at $k=k_0$, k_0' where $[\,,\,]_R$ fails to be so.

For equilibria of type (b), we do not have the inequality (3.1), and the limit (3.16) does not dominate the full expression for T for large $|k|$. For such systems it is possible to construct perturbations g peaked at $v=0$ with arbitrarily large L^2 norm but a finite value of the analogue of $[g, g]_R$. This suggests that these type (b) configurations are inherently more unstable than those of tyep (a).

It is possible to construct equilibrium configurations $g_0(v)$ such that the interval (k_0, k_0') contains subintervals (k_i, k_i') where no solutions to (2.10) exist, so that there are no unstable modes. It would be interesting to examine what quadratic forms are positive-definite in these subintervals of k.

4. Structural instability?

Let us consider again the Hamiltonian (1.1). Besides the value of H there is a second invariant for this system:

$$G=\tfrac{1}{2}\omega_1(p_1^2+q_1^2)+\tfrac{1}{2}\omega_2(p_2^2+q_2^2). \tag{4.1}$$

G is positive, and is the analogue of the quadratic form $[\ ,\ ]_R$ in our problem. What happens when we switch on a nonlinear perturbation linking the two oscillators in (1.1)? One might think that the value of the new Hamiltonian is the only conserved quantity now, and G is a meaningless quantity. But the KAM theorem tells us that if the ratio of frequencies ω_2/ω_1 is sufficiently irrational, then G is deformed to another integral of motion, and the motion still takes place on a two-dimensional torus in the four-dimensional phase space. (The size of the neighbourhoods around the rational numbers that must be avoided go to zero with the strength of the perturbation.)

Is it possible that $[\ ,\ ]_R$ and other members of the family of conserved forms deform to new integrals when nonlinear terms are considered in the evolution equation? The fact that we have an infinite number of degrees of freedom in our system may suggest that this would be difficult; any argument for integrability would have to control resonances from an infinite number of sources. But a closer look at the problem reveals that we do not have a countable infinity of orthogonal degrees of freedom as the independent ones for the linear problem, which would indeed be a complicated case. We have a continuous family of singular Van Kampen modes [12], and the weight of any one mode in expressing a smooth function is zero. Thus only a range of modes carries any weight, and in any range a variety of frequencies are smeared. This could lead to a softening of the resonance structure.

That such is indeed the case is suggested by a recent simulation [9] of the one-dimensional collisionless gravitating system, which is described by an evolution analogous to (2.1), (2.2). It is known that when the equilibrium configuration $f_0(E)$ is a monotonically decreasing function of the energy E then the system is stable due to positivity of the energy quadratic form analogous to (2.11) [13]. If f_0 is not monotonic, pertrubations can be constructed to decrease or keep unchanged the total energy of the system, though the linear theory does not always develop an unstable mode. The simulation in [9] starts with the system far out of equilibrium, and stablizes to an oscillating solution with distribution function not monotonically decreasing with energy, but having instead elongated 'holes' in phase space.

To explain the above in terms of the KAM theorem one observes [10] that only a small measure of orbits resonate with the oscillation; the others distort but remain closed. Because closed curves in this two-dimensional phase space partition the phase space, there is no large drift of points under the oscillation potential.

Similarily, suppose in the plasma case we excite an oscillation periodic in space with period $\lambda = 2\pi/k$ with k outside the range (k_0, k_0'). In considering nonlinear terms we have to consider harmonics nk, in integer, all of which lie outside the range (k_0, k_0'). (The zero mode can presumably be absorbed by redefining the equilibrium configuration.) For a given strength of oscillation, a finite measure of v values would be trapped in resonance regions, while other $v =$ constant orbits would deform but continue to partition phase space, preventing large wanderings of particle orbits. Consequently we do not expect a 'structural instability' due to inclusion of nonlinear terms, in the wavelength domain where no linear instability occurs. (When a growing linear mode arises, the solutions of the linear problem are not 'confined' in a finite region of solution space, as shown by the absence of positive-definite invariants like $[\ ,\ ]_R$ defined on the space of functions. A KAM approach is then not applicable.)

5. Discussion

We have analysed the question: How do we understand the 'anomalous stability' to short wavelength perturbations in equlibria where $g_0(v)$ is not a 'single maximum' distribution? If we ignore the last term in the linearized evolution equation (2.3) then we indeed have an infinite number of manifestly uncoupled degrees of freedom, just like for the two degrees of freedom Hamiltonian (1.1). Is it a specific peculiarity of the form of the integral in (2.3) in that it fails to couple positive energy modes to negative energy modes for sufficiently high wavenumber? As a corollary of our analysis we see that this is not the case. The exact form of the integral term in is not important; any integral term with a sufficiently regular kernel would ensure that this part decreased with $|k|$, and was a compact operator [14] (which is a generalization of the finite rank operator that we had in our case). Similarily, the exact form of the first term in L is not important; all that we used was that (3.1) hold so that this part can contribute dominantly in the form $[\ ,\ ]_R$ for all perturbations of large wavenumber.

It is interesting to see our construction from a mathematical perspective. For $g_0(v)$ a 'single maximum' function, $[\ ,\ ]_Q$ gives a positive-definite conserved inner product on the space of allowed perturbations. If we continue to use $[\ ,\ ]_Q$ as the inner product for all distributions of type (a) (cf section 3) then we get an indefinite inner product space in general. An indefinite space V is called decomposable if we can write $V = V_+ \oplus V_0 \oplus V_-$, with each of these three subspaces orthogonal to the others, and the norm squared of a vector being positive, zero or negative if it belongs to V_+, V_0 or V_- respectively. If V_0 is empty (and the other two parts are complete spaces) then we have a Krein space. Let the distribution function $g_0(v)$ have compact support in v; then the operator L is bounded and we can use the simpler theory of bounded operators in indenfinite metric spaces. On a Krein space we say a bounded operator L is positizable if there exists a real (non-constant) polynomial P such that $P(L)$ is a positive-semidefinite operator ($[g, P(L)g]_Q \geqslant 0$) [15].

In this language, for large wavenumber, we have that the space of allowed perturbations with the norm $[\ ,\ ]_Q$ is a Krein space, and the positive form $[\ ,\ ]_R$ shows that the evolution operator L is positizable. A point ω in the spectrum of a positizable operator is called a critical point if the spectral projection onto every neighbourhood containing ω gives an indefinite subspace. Only at critical points can we have Jordan chains of vectors (as opposed to eigenvectors), and the maximum length of the Jordan chain is two. The spectral theorem for (bounded) positizable operators says that critical points must be contained in the zeros of the positizing polynomial P. These critical points correspond, in our problem, to the frequencies kv_i where the instability can arise. If the complex eigenvalues emerge at these frequencies $\omega = kv_i$, then the onset of instability is characterized by $P(L)$ becoming positive-semidefinite instead of positive-definite†. (The Jordan chain of length two corresponds to the oscillatory mode and the linearly growing mode at the onset of instability.) If the instability arises with $\omega = 0$ then at the corresponding value of k the vector space of perturbations with the inner product $[\,,\,]_Q$ has ceased to be Krein, because the corresponding eigenfunction belongs to the V_0 component of the vector space (cf lemma 3, section 3).

Our approach should extend to plasma perturbations with magnetic field excitations, and to the stellar dynamics case. The computation of higher order perturbation theory corrections to the integral $[\,,\,]_R$, and their relation to the KAM analysis, are under investigation.

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References

[1] Landau L 1946 *J. Phys.* (Moscow) **10** 25
[2] Backus G 1960 *J. Math. Phys.* **1** 178
[3] Case K M 1959 *Ann. Phys.* **7** 349
[4] Li J and Spies G O 1991 *Phys. Fluids* B **3** 1158
[5] C S Gardner 1963 *Phys. Fluids* **6** 839
[6] Morrison P J and Pfirsch D 1990 *Phys. Fluids* B **2** 1105
[7] Cherry T M 1925 *Trans. Camb. Phil. Soc.* **23** 199
[8] Kandrup H E 1991 *Astrophys. J.* **370** 312
[9] Mineau P, Feix M R and Rouet J I 1990 *Astron. Astrophys.* **228** 344
[10] Mathur S D 'Nonlinear oscillations of stellar systems' unpublished
[11] Arnold V I and Avez A 1989 *Ergodic Problems in Classical Mechanics* 1989 (Reading, MA: Addison-Wesley)
[12] Kampen N G 1955 *Physics* **21** 949
[13] Bartholomew P 1971 *Mon. Not. R. Astron. Soc.* **151** 333
[14] Reed M and Simon B 1980 *Methods of Modern Mathematical Physics* (New York: Academic)
[15] Bognar J 1974 *Indefinite Inner Product Spaces* (Berlin: Springer)

† The spectral theorem for bounded positizable operators also tells us that there can be only a finite number of complex eigenvalues, and that the Jordan chains for these must have length one. Thus when k is such that the space of perturbations is Krein we can obtain a stronger result than [2], where such eigenvalues were assumed denumerable, each with a Jordan chain of some length n_j, implying modes of the kind $t^m\, e^{-i\omega_j t}$, $m = 0, 1 \ldots, n_j - 1$.